FINITE DEFORMATIONS OF AN ELASTIC ANISOTROPIC BODY

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The sequential description of an experiment on the homogeneous deformation of a plane specimen is examined. Generalization to the inhomogeneous state results in a closed system of equations connecting the rate of stress and strain change. Problems on the deformation of an elliptical domain with velocities given on the boundary are solved as tests. The material parameters are determined from experiments formulated on the biaxial tension of rubber. The correctness of the proposed model and the possibility of solving problems on large deformations and rotations are shown.

1. We try to construct finite deformation equations by starting just from a physical experiment and the sequential description of its results. We limit ourselves to the plane case. We take a square specimen of homogeneous material of dimension $\ell_0 \times \ell_0$ (such as a rubber sheet). We apply identical uniformly distributed forces in pairs to opposite sides. The specimen consequently changes its size and takes the shape of a parallelogram. We measure the new values of the side lengths ℓ_1, ℓ_2, the angle between them $(\pi/2 - \delta)$, and the acting forces in the experiment. For definiteness we represent the latter in the form of two components directed along the normal and along the sides (Fig. 1). Therefore, the directly measurable experimental data permit determination of four functions

$$P_{ij} = F_{ij}(l_1, l_2, \delta, l_0), \quad i, j = 1, 2. \tag{1.1}$$

Because of the homogeneity of the material and the uniformity of the load distribution, the dependence on the last argument has the form

$$P_{ij} = l_0 F_{ij}(l_1/l_0, l_2/l_0, \delta). \tag{1.2}$$

Here it is already assumed that the material behavior is independent of the loading history, the F_{ij} are functions not functionals.

Since all the forces, and initial and new specimen dimensions are known from test, then any combination of the measured quantities, for instance

$$\sigma_{11}^0 = P_{11}/l_2, \quad \sigma_{12}^0 = P_{12}/l_2, \quad \sigma_{21}^0 = P_{21}/l_1, \quad \sigma_{22}^0 = P_{22}/l_1, \quad E_1 = l_1/l_0, \quad E_2 = l_2/l_0. \tag{1.3}$$

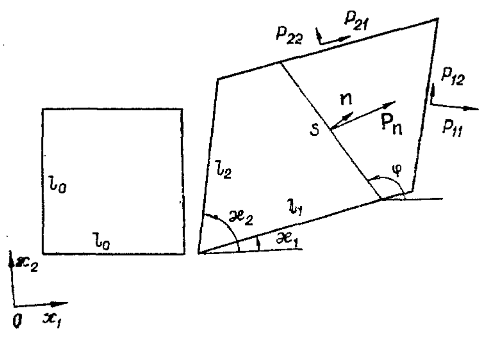

Fig. 1

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can be used during processing the results. The ratio ℓ_i/ℓ_0 and the angle δ can be considered loading parameters dependent on the time t. We denote the rate of change with respect to t by a dot. The quantities $\dot{\ell}_1$ and $\dot{\ell}_2$, as also ℓ_1 and ℓ_2, are measured directly in experiment (or are given by the test program). Consequently, we also use the ratio $\dot{\ell}_i/\ell_i$ for the processing:

$$\dot{l}_i/l_i = \dot{l}_i/l_0 \cdot l_0/l_i = \dot{E}_i/E_i = \dot{\varepsilon}_i^0 \quad (\varepsilon_i^0 = \ln E_i). \tag{1.4}$$

The dependences (1.1)-(1.3) yield

$$\begin{aligned} \dot{\sigma}_{1i}^0 &= \frac{E_1 \partial F_{1i}}{E_2 \partial E_1} \dot{\varepsilon}_1^0 + \left(\frac{\partial F_{1i}}{\partial E_2} - \sigma_{1i}^0\right) \dot{\varepsilon}_2^0 + \frac{1}{E_2} \frac{\partial F_{1i}}{\partial \delta} \dot{\delta}, \\ \dot{\sigma}_{2i}^0 &= \left(\frac{\partial F_{2i}}{\partial E_1} - \sigma_{2i}^0\right) \dot{\varepsilon}_1^0 + \frac{E_2 \partial F_{2i}}{E_1 \partial E_2} \dot{\varepsilon}_2^0 + \frac{1}{E_1} \frac{\partial F_{2i}}{\partial \delta} \dot{\delta}. \end{aligned} \tag{1.5}$$

Let us introduce a laboratory Cartesian coordinate system Ox_1x_2. We denote the velocity components of specimen points by v_1, v_2. Because of homogeneity they depend on the coordinates $v_i = \alpha_{ij}x_j$. It is more convenient to write the coefficients α_{ij} in the form of derivatives $v_{i,j}$. We imaginarily dissect the specimen by a certain line with normal $\mathbf{n}$ and denote the vector of the force with which one part of the specimen acts on the other by $\mathbf{P}_n$; s is the length of the demarcation line; σ_{n1} and σ_{n2} are components of the vector $\boldsymbol{\sigma}_n = \mathbf{P}_n/s$, φ, $\varkappa_1$, $\varkappa_2$ are, respectively, the slope of the normal $\mathbf{n}$ and the sides of the specimen to the Ox_1 axis (see Fig. 1). It is evident from equilibrium conditions that the vector $\boldsymbol{\sigma}_n$ is independent of the location of the dissecting line. Its projections are

$$\begin{aligned} \sigma_{n1} &= \frac{\cos(\varphi - \varkappa_1)}{\sin(\varkappa_2 - \varkappa_1)}(\sigma_{11}^0 \sin\varkappa_2 + \sigma_{12}^0 \cos\varkappa_2) + \frac{\cos(\varphi - \varkappa_2)}{\sin(\varkappa_2 - \varkappa_1)}(\sigma_{21}^0 \cos\varkappa_1 - \sigma_{22}^0 \sin\varkappa_1), \\ \sigma_{n2} &= \frac{\cos(\varphi - \varkappa_1)}{\sin(\varkappa_2 - \varkappa_1)}(-\sigma_{11}^0 \cos\varkappa_2 + \sigma_{12}^0 \sin\varkappa_2) + \frac{\cos(\varphi - \varkappa_2)}{\sin(\varkappa_2 - \varkappa_1)}(\sigma_{21}^0 \sin\varkappa_1 + \sigma_{22}^0 \cos\varkappa_1). \end{aligned}$$

We replace the subscript n for the normal $\mathbf{n}$ directed along the Ox_i axis by i. The components σ_{ij} form a Cauchy stress tensor

$$\begin{aligned} \sigma_{11} &= \frac{1}{\sin(\varkappa_2 - \varkappa_1)}[\sigma_{11}^0 \cos\varkappa_1 \sin\varkappa_2 + (\sigma_{12}^0 + \sigma_{21}^0)\cos\varkappa_1 \cos\varkappa_2 - \sigma_{22}^0 \sin\varkappa_1 \cos\varkappa_2], \\ \sigma_{12} &= \frac{1}{\sin(\varkappa_2 - \varkappa_1)}[(\sigma_{22}^0 - \sigma_{11}^0)\cos\varkappa_1 \cos\varkappa_2 + \sigma_{12}^0 \cos\varkappa_1 \sin\varkappa_2 + \sigma_{21}^0 \sin\varkappa_1 \cos\varkappa_2], \\ \sigma_{21} &= \frac{1}{\sin(\varkappa_2 - \varkappa_1)}[(\sigma_{11}^0 - \sigma_{22}^0)\sin\varkappa_1 \sin\varkappa_2 + \sigma_{12}^0 \sin\varkappa_1 \cos\varkappa_2 + \sigma_{21}^0 \cos\varkappa_1 \sin\varkappa_2], \\ \sigma_{22} &= \frac{1}{\sin(\varkappa_2 - \varkappa_1)}[-\sigma_{11}^0 \sin\varkappa_1 \cos\varkappa_2 + (\sigma_{12}^0 + \sigma_{21}^0)\sin\varkappa_1 \sin\varkappa_2 + \sigma_{22}^0 \cos\varkappa_1 \sin\varkappa_2]. \end{aligned} \tag{1.6}$$

The forces P_{ij} and their changes are related by the condition that there is no rotational moment. These conditions appear simplest in the terminology of σ_{ij}: $\sigma_{12} = \sigma_{21}$, $\dot{\sigma}_{12} = \dot{\sigma}_{21}$. From kinetic definitions there follows

$$\begin{aligned} \dot{\varepsilon}_j^0 &= \frac{v_{1,1} + v_{2,2}}{2} + \frac{v_{1,1} - v_{2,2}}{2} \cos 2\varkappa_j + \frac{v_{1,2} + v_{2,1}}{2} \sin 2\varkappa_j, \\ \dot{\varkappa}_j &= \frac{v_{2,1} - v_{1,2}}{2} + \frac{v_{2,2} - v_{1,1}}{2} \sin 2\varkappa_j + \frac{v_{1,2} + v_{2,1}}{2} \cos 2\varkappa_j, \quad \dot{\delta} = \dot{\varkappa}_1 - \dot{\varkappa}_2. \end{aligned} \tag{1.7}$$

Let us go over to velocities in (1.6). Taking account of the expressions for $\dot{\sigma}_{ij}^0$ from (1.5) and of $\dot{\varepsilon}_j^0$ and $\dot{\delta}$ from (1.7), we obtain

$$\begin{aligned} \dot{\sigma}_{11} + \sigma_{12}(v_{2,1} - v_{1,2}) &= B_{11}v_{1,1} + B_{12}v_{2,2} + B_{13}\frac{v_{1,2} + v_{2,1}}{2}, \\ \dot{\sigma}_{22} - \sigma_{12}(v_{2,1} - v_{1,2}) &= B_{21}v_{1,1} + B_{22}v_{2,2} + B_{23}\frac{v_{1,2} + v_{2,1}}{2}, \\ \dot{\sigma}_{12} - \frac{\sigma_{11} - \sigma_{22}}{2}(v_{2,1} - v_{1,2}) &= B_{31}v_{1,1} + B_{32}v_{2,2} + B_{33}\frac{v_{1,2} + v_{2,1}}{2}, \end{aligned} \tag{1.8}$$

where

$$B_{1k}=\frac{\cos\varkappa_1\sin\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{11}-\frac{\sin\varkappa_1\cos\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{22}+\frac{\cos\varkappa_1\cos\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{12}, \tag{1.9}$$

$$B_{2k}=-\frac{\sin\varkappa_1\cos\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{11}+\frac{\cos\varkappa_1\sin\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{22}+\frac{\sin\varkappa_1\sin\varkappa_2}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{12},$$

$$B_{3k}=\frac{1}{2}\frac{\cos(\varkappa_1+\varkappa_2)}{\sin(\varkappa_2-\varkappa_1)}(\Phi^k_{22}-\Phi^k_{11})+\frac{1}{2}\frac{\sin(\varkappa_1+\varkappa_2)}{\sin(\varkappa_2-\varkappa_1)}\Phi^k_{12},\quad k=1,2,3,$$

$$\Phi^1_{11}=\frac{E_1\partial F_{11}}{E_2\partial E_1}\cos^2\varkappa_1+\left(\frac{\partial F_{11}}{\partial E_2}-\sigma^0_{11}\right)\cos^2\varkappa_2-\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}\sin\varkappa_1\cos\varkappa_1+$$
$$+\left(\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}+2\sigma^0_{12}\right)\sin\varkappa_2\cos\varkappa_2,$$

$$\Phi^2_{11}=\frac{E_1}{E_2}\frac{\partial F_{11}}{\partial E_1}\sin^2\varkappa_1+\left(\frac{\partial F_{11}}{\partial E_2}-\sigma^0_{11}\right)\sin^2\varkappa_2+\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}\sin\varkappa_1\cos\varkappa_1-$$
$$-\left(\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}+2\sigma^0_{12}\right)\sin\varkappa_2\cos\varkappa_2,$$

$$\Phi^3_{11}=\frac{E_1}{E_2}\frac{\partial F_{11}}{\partial E_1}\sin 2\varkappa_1+\left(\frac{\partial F_{11}}{\partial E_2}-\sigma^0_{11}\right)\sin 2\varkappa_2+\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}\cos 2\varkappa_1-$$
$$-\left(\frac{1}{E_2}\frac{\partial F_{11}}{\partial\delta}+2\sigma^0_{12}\right)\cos 2\varkappa_2,$$

$$\Phi^1_{22}=\left(\frac{\partial F_{22}}{\partial E_1}-\sigma^0_{22}\right)\cos^2\varkappa_1+\frac{E_2}{E_1}\frac{\partial F_{22}}{\partial E_2}\cos^2\varkappa_2-\left(\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}+2\sigma^0_{21}\right)\sin\varkappa_1\cos\varkappa_1+$$
$$+\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}\sin\varkappa_2\cos\varkappa_2,$$

$$\Phi^2_{22}=\left(\frac{\partial F_{22}}{\partial E_1}-\sigma^0_{22}\right)\sin^2\varkappa_1+\frac{E_2}{E_1}\frac{\partial F_{22}}{\partial E_2}\sin^2\varkappa_2+\left(\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}+2\sigma^0_{21}\right)\sin\varkappa_1\cos\varkappa_1-$$
$$-\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}\sin\varkappa_2\cos\varkappa_2,$$

$$\Phi^3_{22}=\left(\frac{\partial F_{22}}{\partial E_1}-\sigma^0_{22}\right)\sin 2\varkappa_1+\frac{E_2}{E_1}\frac{\partial F_{22}}{\partial E_2}\sin 2\varkappa_2+\left(\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}+2\sigma^0_{21}\right)\cos 2\varkappa_1-$$
$$-\frac{1}{E_1}\frac{\partial F_{22}}{\partial\delta}\cos 2\varkappa_2,$$

$$\Phi^1_{12}=-\sigma^0_{12}-\sigma^0_{21}+\left(\frac{E_1}{E_2}\frac{\partial F_{12}}{\partial E_1}+\frac{\partial F_{21}}{\partial E_1}+\sigma^0_{21}\right)\cos^2\varkappa_1+$$
$$+\left(\frac{\partial F_{12}}{\partial E_2}+\frac{E_2}{E_1}\frac{\partial F_{21}}{\partial E_2}+\sigma^0_{12}\right)\cos^2\varkappa_2-$$
$$-\left(\frac{1}{E_2}\frac{\partial F_{12}}{\partial\delta}+\frac{1}{E_1}\frac{\partial F_{21}}{\partial\delta}\right)(\sin\varkappa_1\cos\varkappa_1-\sin\varkappa_2\cos\varkappa_2),$$

$$\Phi^2_{12}=-\sigma^0_{12}-\sigma^0_{21}+\left(\frac{E_1}{E_2}\frac{\partial F_{12}}{\partial E_1}+\frac{\partial F_{21}}{\partial E_1}+\sigma^0_{21}\right)\sin^2\varkappa_1+$$
$$+\left(\frac{\partial F_{12}}{\partial E_2}+\frac{E_2}{E_1}\frac{\partial F_{21}}{\partial E_2}+\sigma^0_{12}\right)\sin^2\varkappa_2+$$
$$+\left(\frac{1}{E_2}\frac{\partial F_{12}}{\partial\delta}+\frac{1}{E_1}\frac{\partial F_{21}}{\partial\delta}\right)(\sin\varkappa_1\cos\varkappa_1-\sin\varkappa_2\cos\varkappa_2),$$

$$\Phi^3_{12}=\left(\frac{E_1}{E_2}\frac{\partial F_{12}}{\partial E_1}+\frac{\partial F_{21}}{\partial E_1}+\sigma^0_{21}\right)\sin 2\varkappa_1+\left(\frac{\partial F_{12}}{\partial E_2}+\frac{E_2}{E_1}\frac{\partial F_{21}}{\partial E_2}+\sigma^0_{12}\right)\sin 2\varkappa_2+$$
$$+\left(\frac{1}{E_2}\frac{\partial F_{12}}{\partial\delta}+\frac{1}{E_1}\frac{\partial F_{21}}{\partial\delta}\right)(\cos 2\varkappa_1-\cos 2\varkappa_2).$$

The Jaumann derivatives of the stress tensor [1] are here extracted in the left side. The coefficients B_{ij} in system (1.8) are instantaneous elastic moduli of the material in this state. For an isotropic material their expression for the stress tensor in the principal axes is obtained in [2, 3]. It should be emphasized that (1.1) and (1.8) and (1.9) are the identical relationships describing the finite specimen behavior under arbitrary deformation in different terminologies. Additional hypotheses were not introduced in the derivation of (1.8) and (1.9) and only new natural definitions and data of the macrotest were utilized. Such a more complex mode of writing (1.1) was required in connection with the fact that it allows generalization to the inhomogeneous state.

Let there be an arbitrary stress distribution in a deformable body. Let us isolate a sufficiently small element within whose limits the velocity distribution can be considered linear while the stress state is homogeneous. All the constructions examined above are valid (gradient models are excluded). The state parameters E_i, $\varepsilon_i{}^0$, $\varkappa_i$, δ, σ_{ij} , and σ_{ij} here become functions of the coordinates x_1, x_2. Since the derivative with respect to the time, denoted by a dot, is referred to a fixed material element, then the total derivative $\partial/\partial t + v_i \partial/\partial x_i$, where $\partial/\partial t$ is the partial derivative for fixed coordinates x_i as usual, corresponds to it in the inhomogeneous case. Together with the equilibrium equations

$$\frac{\partial}{\partial x_1}\frac{\partial \sigma_{11}}{\partial t} + \frac{\partial}{\partial x_2}\frac{\partial \sigma_{12}}{\partial t} = 0, \quad \frac{\partial}{\partial x_1}\frac{\partial \sigma_{12}}{\partial t} + \frac{\partial}{\partial x_2}\frac{\partial \sigma_{22}}{\partial t} = 0 \qquad (1.10)$$

the relationships (1.8) and (1.9) form a closed system in the five unknowns: $\partial\sigma_{11}/\partial t$, $\partial\sigma_{12}/\partial t$, $\partial\sigma_{22}/\partial t$, v_1, and v_2. Its solution yields the change of state of the deformable body within one small loading step. The parameters σ_{ij}, $\sigma_{ij}{}^0$, $\varkappa_i$, and E_i characterizing the state already achieved are knowns in the system. After the problem has been solved, their new values are converted by means of the formulas (1.4), (1.5), (1.7), and (1.8). Therefore, the system (1.8)-(1.10) and formulas (1.4), (1.5), and (1.7) permit solution of the problem of arbitrary deformations of an anisotropic elastic body.

Furthermore, (1.1) should satisfy constraints of general nature whose formulation permits reduction of the class of necessary experiments in their determination. The first constraint of this nature is energetic: the work should equal zero on any closed deformation path. Let us examine such paths in the neighborhood of a certain homogeneous state ($\sigma_{ij} = \sigma_{ij}{}^c$) when a change in the elastic coefficients B_{km} (k, m = 1, 2, 3) in (1.8) can be neglected. For $0 \le t \le 1$ let the velocity distribution be

$$v_i = \dot{x}_i = h\varphi_{ij}(t)x_j, \; h \ll 1. \qquad (1.11)$$

The requirement that the contour $x_i(0) = x_i(1)$ be closed yields

$$\Phi_{ij}(1) + h\int_0^1 \varphi_{im}\Phi_{mj}dt = O(h^2) \quad \left(\Phi_{ij}(t) = \int_0^t \varphi_{ij}(\tau)\,d\tau\right). \qquad (1.12)$$

Considering (1.8) and (1.11) as ordinary differential equations, we expand their solution in a small parameter h and compose an expression for the work of the external forces $W = \int_0^1 \sigma_{ij} v_{i,j} V dt$ [$V = V_0(1 + h\Phi_{11} + h\Phi_{22}) + O(h^2)$ is the volume]. The conditions that this work equal zero for any functions $\varphi_{ij}(t)$ satisfying (1.12) are of the form $B_{12} + \sigma_{11} = B_{21} + \sigma_{22}$, $B_{31} + \sigma_{12} = B_{13}/2$, $B_{32} + \sigma_{12} = B_{23}/2$ (because of the arbitrariness of the initial stresses $\sigma_{ij}{}^c$ the superscript c can be omitted). Taking account of the last equalities, relations (1.8) can be rewritten as

$$\begin{aligned}
&\dot{\sigma}_{11} - \sigma_{12}\left(\frac{3}{2}v_{1,2} - \frac{1}{2}v_{2,1}\right) + \sigma_{11}v_{2,2} = \\
&\quad = K_{11}v_{1,1} + K_{12}v_{2,2} + K_{13}(v_{1,2} + v_{2,1}), \\
&\dot{\sigma}_{22} + \sigma_{12}\left(\frac{1}{2}v_{1,2} - \frac{3}{2}v_{2,1}\right) + \sigma_{22}v_{1,1} = K_{12}v_{1,1} + K_{22}v_{2,2} + K_{23}(v_{1,2} + v_{2,1}), \\
&\dot{\sigma}_{12} + \frac{\sigma_{11}}{2}\left(\frac{1}{2}v_{1,2} - \frac{3}{2}v_{2,1}\right) - \frac{\sigma_{22}}{2}\left(\frac{3}{2}v_{1,2} - \frac{1}{2}v_{2,1}\right) + \frac{\sigma_{12}}{2}(v_{1,1} + v_{2,2}) = \\
&\quad = K_{13}v_{1,1} + K_{23}v_{2,2} + K_{33}(v_{1,2} + v_{2,1})
\end{aligned} \qquad (1.13)$$

(K_{11} ... K_{33} are known combinations of B_{km} and σ_{ij}).

When going over from (1.8) to (1.13), components of the symmetric tensor that is the objective derivative of the stress were extracted to the left side in a natural way. The

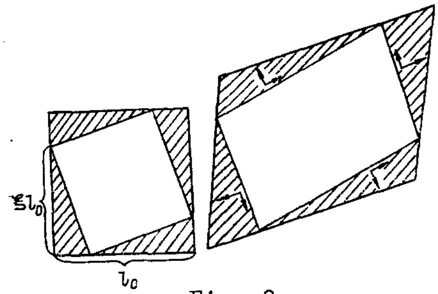

Fig. 2

potentiality condition is simply the symmetry condition of a fourth-rank tensor that connects it to the strain rates. A measure of the stress change for which the matrix of instantaneous elastic moduli is symmetric is also introduced in [2]. The components of this velocity differ from the left side of (1.13) by the components $(1/2)(\sigma_{ik}v_{k,j} + \sigma_{jk}v_{k,i})$.

The second constraint of general nature refers to the case when the medium is isotropic. Let us isolate a "new" specimen in the specimen in the initial state, a square of smaller dimensions defined by the parameter $0 \le \xi \le 1$ (Fig. 2). We consider the shaded triangles as part of the loading unit. Then the equilibrium and deformation homogeneity conditions permit finding the force applied to the "new" specimen and all its geometric characteristics, too. Because of the isotropy they should be related by the same dependences (1.1) for any $0 \le \xi \le 1$. This yields four functional equations in the four functions F_{ij}. Their general solution is found successfully by using the following circumstance.

For any fixed deformation of the original specimen achieved at the time $t = t^0$ the free parameter ξ can be selected such that the appropriate "new" specimen would become rectangular at $t = t^0$. The converse is also true: if we limit ourselves just to a loading class without shear for the original specimen, then by using arbitrary ξ it is always possible to find such a "new" specimen that would experience any previously assigned deformation. This means that to obtain the complete characteristics of an isotropic material it is sufficient to perform experiments of just biaxial tension. It is afterwards necessary to express the functions F_{ij} $(.,.,\delta)$ in terms of F_{ij} $(.,.,0)$ and substitute into (1.9). This road is fraught with awkward calculations. The same final result is obtained if we were to go from the relationships describing the biaxial loading test over to equations of the type (1.13) directly. For the test interpretation it is more convenient to take $k_i = \ell_0{}^2/\ell_i{}^2 = E_i{}^{-2}$ as arguments

$$\sigma_{11}^0 = G(k_1, k_2), \quad \sigma_{22}^0 = G(k_2, k_1), \quad \sigma_{12}^0 = \sigma_{21}^0 \equiv 0. \tag{1.14}$$

Let the $Ox_1{}'x_2{}'$ coordinate system whose axes are directed along the sides of the specimen be rotated through an angle α relative to the laboratory coordinates. Then (1.6) will yield

$$\begin{aligned}\sigma_{11} &= G(k_1, k_2)\cos^2\alpha + G(k_2, k_1)\sin^2\alpha, \ \sigma_{22} = G(k_1, k_2)\sin^2\alpha + \\ &+ G(k_2, k_1)\cos^2\alpha, \ \sigma_{12} = [G(k_1, k_2) - G(k_2, k_1)]\sin\alpha\cos\alpha.\end{aligned} \tag{1.15}$$

The problem of describing complex recharging occurs here. If the specimen is stretched along the $Ox_1{}'$ and $Ox_2{}'$ axes to the values $k_1(t_0)$ and $k_2(t_0)$ at the time t and it experiences an arbitrary homogeneous recharging characterized by small displacements $v_i dt$, then it will cease to be rectangular in the general case. Since the state of the specimen under consideration emerges beyond the biaxial loading test framework, it is impossible to use the relationships (1.14) at the next time $t^0 + dt$ directly. This difficulty is overcome successfully by using the "new" specimen concept. As mentioned above, this latter can always be selected such that it is deformed into a rectangle at the time $t^0 + dt$. Then its orientation and size determine the values $\alpha(t^0 + dt)$, $k_1(t^0 + dt)$, $k_2(t^0 + dt)$. Therefore, as functions of time, α and k_i describe not the evolution of the state of some element consisting of identical material particles but the deformation process as a whole,

$$\begin{aligned}\dot{k}_1 &= -2k_1[v_{1,1}\cos^2\alpha + (v_{1,2} + v_{2,1})\sin\alpha\cos\alpha + v_{2,2}\sin^2\alpha], \\ \dot{k}_2 &= -2k_2[v_{1,1}\sin^2\alpha - (v_{1,2} + v_{2,1})\sin\alpha\cos\alpha + v_{2,2}\cos^2\alpha], \\ \dot{\alpha} &= \frac{v_{2,1} - v_{1,2}}{2} + \frac{k_1 + k_2}{k_1 - k_2}\left[\frac{v_{1,1} - v_{2,2}}{2}\sin 2\alpha - \frac{v_{1,2} + v_{2,1}}{2}\cos 2\alpha\right].\end{aligned} \tag{1.16}$$

If (1.15) is differentiated, then by using (1.16) it can be reduced to the form (1.13). The following values of the elastic moduli are utilized here:

$$K_{11} = Q_1 \frac{\sin^2 2\alpha}{2} + Q_2 \cos 2\alpha + Q_3, \quad K_{12} = -Q_1 \frac{\sin^2 2\alpha}{2} + H,$$
$$K_{22} = Q_1 \frac{\sin^2 2\alpha}{2} - Q_2 \cos 2\alpha + Q_3, \quad K_{33} = -Q_1 \frac{\sin^2 2\alpha}{2} + Q_4, \tag{1.17}$$
$$K_{13} = [-Q_1 \cos 2\alpha + Q_2] \frac{\sin 2\alpha}{2}, \quad K_{23} = [Q_1 \cos 2\alpha + Q_2] \frac{\sin 2\alpha}{2},$$

where

$$Q_1 = -\frac{k_1 + k_2}{k_1 - k_2}[G(k_1, k_2) - G(k_2, k_1)] - k_1 \frac{\partial G(k_2, k_1)}{\partial k_1} + k_2 \frac{\partial G(k_2, k_1)}{\partial k_2} - Q_2,$$
$$Q_2 = -k_1 \frac{\partial G(k_1, k_2)}{\partial k_1} + k_2 \frac{\partial G(k_2, k_1)}{\partial k_2}, \quad Q_3 = -k_1 \frac{\partial G(k_1, k_2)}{\partial k_1} - k_2 \frac{\partial G(k_2, k_1)}{\partial k_2}, \quad Q_4 = \tag{1.18}$$
$$= \left(\frac{1}{4} - \frac{1}{2} \frac{k_1 + k_2}{k_1 - k_2}\right) G(k_1, k_2) + \left(\frac{1}{4} + \frac{1}{2} \frac{k_1 + k_2}{k_1 - k_2}\right) G(k_2, k_1), \quad H = G(k_1, k_2) - 2k_2 \frac{\partial G(k_1, k_2)}{\partial k_2}.$$

The condition for existence of a potential that has the form $H(k_1, k_2) = H(k_2, k_1)$ in this case is used here.

Relationships (1.17) show that the moduli K_{13}, $K_{23} \neq 0$ in the general case, i.e., in the presence of initial stresses the medium will behave as an anisotropic medium with respect to small recharging although it is isotropic "in the large." This fact can be treated as an anisotropy induced by stresses existing in the medium [3, 4] in contrast to the "inherent" anisotropy described by (1.8) and (1.9).

Therefore, for an isotropic elastic body, the closed system of equation in $\partial\sigma_{11}/\partial t$, $\partial\sigma_{22}/\partial t$, $\partial\sigma_{12}/\partial t$, v_1, and v_2 has the form (1.10), (1.13), (1.17), and (1.18). Conversion of the parameters σ_{ij}, k_i, and α in the coefficients of this system in the next step is realized by means of (1.13) and (1.16).

2. Biaxial tension tests on rubber under plane stress state conditions were composed to determine the function G. A specimen of size 120 × 120 mm and 0.4 mm thickness was arranged horizontally and stretched by seven clamps along each side. To assure that the clamps were parallel, there was the possibility of displacing them along the specimen sides varying in length. The deformations were measured at the middle of the specimen at an 80-mm base. The forces P_1 and P_2 were transmitted through modules from the suspended loads. The maximal deformation $(E_1 - 1)$ was 60% in the test. The rubber disclosed the property of creep. Consequently, not less than a 2-min hold was made before each measurement. This was sufficient for the creep deformation to be reduced to the 1-2% measurement error.

The simplest kind of function G satisfying conditions for the existence of an elastic potential, isotropy, and incompressibility, a Mooney diagram, was taken as the basis for processing the results:

$$G(k_1, k_2) = 2\sqrt{k_1 k_2}\left[C_1\left(\frac{1}{k_1} - k_1 k_2\right) + C_2\left(\frac{1}{k_1 k_2} - k_2\right)\right] \tag{2.1}$$

(the material constants C_1 and C_2 were computed by least squares on the basis of 52 measurements: C_1 = 155 N/m, C_2 = 7.23 N/m). The rms residual of (2.1) here equals 1.34 N/m, which is at the level of the measurement error relative to the characteristic values of σ_1 and σ_2.

3. A series of numerical computations was performed to verify the theoretical constructions and the algorithm developed. A dimensionless problem on the deformation of an elliptical domain with the semiaxes $a = 1/\lambda$, $b = 1$, $\lambda < 1$ (Fig. 3) was solved as a test. This problem was examined in detail in [5]. Both components of the velocity vector $\mathbf{v}(x_1, x_2)$ were given on the boundary. The vector is directed along the tangent to the boundary and was determined either by equality A: $|\mathbf{v} \times \mathbf{r}| = 1$ (Kepler inversion law) or by equality B: $|\mathbf{v}| = 1$ ($\mathbf{r}$ is the radius-vector drawn from the center of the ellipse). There are no initial stresses in the original state. After a specific time Tn (n = 1, 2, ...) all the boundary points return to their original position. For problem A, $T = 2\pi/\lambda$, while for B, T = L (L is the length of the boundary). The necessary condition for correctness of the model and

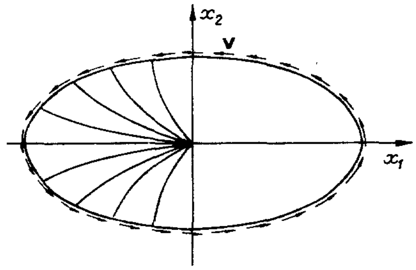

Fig. 3

the algorithm is satisfaction of the following requirements: all the interior points should return to their original position after a time T in both problems; all the stress components should vanish after the time Tn/2; material points initially on the axes should pass through the axes at the times t = Tn/4; the distribution of the stress, deformation, and their rates (in the sense of any definitions) should always be homogeneous in problem A [6]. Computations were performed for domains with the semiaxes a = 2, 1.6, 1.2, and b = 1. The finite element method with linear interpolation of the displacements was used. The problem (1.10), (1.13), (1.17), (1.18) was solved at each loading step, after which the stresses and deformations on the whole domain were converted. The stiffness matrix is obtained symmetric when using the governing equations in the form (1.13). Satisfaction of this property, as well as the equilibrium equations, was periodically checked numerically during solution of the problem.

The system in displacements was solved by the method of sequential upper relaxation with an accelerating factor 1.4-1.6. Taken as initial approximation was the solution in the previous step, here a 10^{-5} accuracy in the uniform metric was usually reached after 30-50 iterations.

To increase the accuracy of the approximation, a predictor-corrector scheme was used. When solving problem B, the loading step $\Delta t = T/400$ and domain discretization into 310 stress elements for t = T/2 was zero-set with 0.6% accuracy relative to the maximal that appeared during the whole loading time. Successive stages in the deformation of the material curve, that coincided with the ellipse minor semiaxis at the initial time, are shown in Fig. 3. Experiments were formulated according to the method in [5, 6] for rubber and showed satisfactory agreement with analysis.

The distribution of the stress, the deformation and their rates is obtained homogeneously in problem A. For t = T/2 the stresses returned to zero values with up to 0.7% accuracy.

Problem B was solved for comparison within the framework of equations that do not take account of the induced material anisotropy: $B_{13} = B_{23} = B_{31} = B_{32} = 0$, $B_{11} = B_{22} = \lambda + 2\mu$, $B_{12} = B_{21} = \lambda$, and $B_{33} = 2\mu$ (λ and μ are Lamé constants). The stiffness matrix is obtained nonsymmetrically here. All the interior points return to the original positions, but the stresses do not vanish for t = T/2 and are 10-15% of the maximal values.

Two fundamental approaches are used at this time to solve problems with large deformations. The first starts from finite relationships between the selected measures of the stresses and deformations and operates with substantially nonlinear equations [7-9]. Lagrange coordinates are ordinarily utilized here since this permits solution of the problem for a domain with known boundaries.

The whole process of body deformation from the initial into the final configuration is considered in the second (incremental) approach. It is a sequence of small recharging steps within the limits of each of which the response of the medium to the external action can be considered linear while the domain occupied by the body is unchanged. This permits reduction of the original nonlinear problem to a sequence of linear problems with known boundaries. Moreover, such a method of description is natural even from the mechanical point of view since the loading process is really always realized gradually and the deformable body passes through the whole sequence of intermediate states between the initial and the final. This approach is possible both in Lagrange [10-13] and Euler [14] coordinates. This paper is executed by using Euler coordinates within the framework of the second approach.

Thus, the sequential description of an experiment on homogeneous loading results in a natural closed model that predicts the appearance of induced anisotropy, which is necessary for correctly taking account of complex loading effects. The model and algorithm satisfy the formulated test requirements and agree with test and permit solution of the problem on large deformations of an elastic anisotropic body.

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